An assign-and-route matheuristic for the time-dependent Inventory Routing Problem

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Abstract

In this paper, we consider a variant of the Inventory Routing Problem (IRP), the Time-Dependent IRP (TD-IRP). The TD-IRP extends the routing component of the IRP by making the travelling time between two locations no longer constant but depending on the departure time. In order to investigate the relevance of considering time-dependent travelling time functions, a set of new benchmark instances based on real-data is assumed. Numerical experiments show that optimising with time-dependent travelling times is cost-efficient, but computationally challenging. Thus, we propose a matheuristic that decomposes the problem, based on the observation of the structure of optimal TD-IRP solutions. The proposed matheuristic defines the set of clients to visit and the quantity to deliver for each period first and solves the routing problem second. Numerical experiments prove it to be very efficient and yield solutions with small gaps to the best lower bounds found. Because it separates the routing problem, the proposed matheuristic opens the possibility to solve the TD-IRP very efficiently by taking advantage of the rich literature on time-dependent routing problems.

Keywords: Logistics; inventory routing problem; time-dependent travelling time; realistic benchmark; matheuristic

1 Introduction

The Inventory Routing Problem (IRP) is the integration of two sub-problems of the supply chain: inventory management and transportation. It emerged in the context of Vendor Managed Inventory (VMI), where the inventories of the clients are controlled by the supplier. The objective of the IRP is to decide, for each period of the time horizon, how much quantity to send to each client and following which sequences the clients must be visited, minimising the total costs incurred by both inventory holding and transportation while satisfying a set of constraints.

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During the last decades, the IRP literature has received a great deal of attention (Coelho et al., 2014). In the pursuit of models that are more and more representative of real-life situations, many variants emerged to cater for the requirements of the actors involved and the challenges that real-life situations impose. A few examples of these variants are: the IRP with time windows, where a client requires to be visited only within a time interval; travel-time constrained IRP, where tours must be completed before a certain time; IRP with transshipment, where one client can be replenished from the inventory of another client rather than from the supplier... Another feature that real-life situations impose is the volatility of the data. In the IRP, different parameters are subject to data uncertainty, such as the demand of the clients or the travelling time between two locations, especially in urban logistics, where traffic conditions are very uncertain or when handling products that are perishable or clients with time windows. In the literature, this problem is handled by considering stochastic parameters and solved through stochastic and robust optimisation approaches (Rahimi et al., 2017; Lefever et al., 2021; Rodrigues et al., 2019). However, when it comes to travelling time, a deterministic way of taking into consideration this volatility is to consider it as time-dependent.

Time-dependent problems consider that the travelling time between two locations does not depend only on the departure and arrival locations, but depends on the time of departure as well. The literature of time-dependent routing problems is quite rich (Gendreau et al., 2015) and shows that time-dependent problems tend to be more complex than their basic counterparts, so the literature is mainly focused on how to solve these problems more efficiently. To the best of our knowledge, these problems have mostly been considered for pure routing problems such as the Time-Dependent Travelling Salesman Problem (TD-TSP) or the Time-Dependent Vehicle Routing Problem (TD-VRP), and only once for integrated problems such as IRP. As the time-dependent aspect can be more representative of the volatility of real-life traffic conditions than considering the travelling time as an uncertain quantity, we believe that the TD-IRP is worth investigating.

In this paper, we therefore consider a variant of the IRP: the Time-Dependent Inventory Routing Problem (TD-IRP) with travel-time constraints, where the travelling times between locations are time-dependent and the length of tours is time-constrained. This paper contributes to the literature by generating a new benchmark set for the TD-IRP, based on real-life transportation data collected through 6 years in the city of Lyon. Moreover, it investigates the relevance of considering time-dependent travelling times for the IRP through computational experiments. Finally, it proposes a matheuristic that can be used to solve large-sized instances.

The paper is presented as follows: Section 2 reviews the literature of the IRP, in general, and the IRP when travelling times are the main focus, in particular. It shows that the volatility of the travelling time in the IRP literature is mostly handled by considering it as uncertain data. As a result, we turn to pure routing problems to investigate how the time-dependent aspect is handled. In Section 3 the TD-IRP is described through a mathematical formulation and an illustrative example. In Section 4 a new realistic benchmark for the TD-IRP is generated and the process of generation described. In Section 5, the relevance of considering time-dependent travelling times for the IRP is investigated. To that purpose, an exact branch-and-cut algorithm is presented, and a comparison between optimal time-dependent solutions and optimal constant travel-time solutions when re-solved in a time-dependent environment is proposed. The results show that considering time-dependent travelling times can be beneficial cost-wise. However, it also shows that solving TD-IRPs is extremely difficult, thus the need for new efficient algorithms to solve them. Section 6 presents a matheuristic and discusses the results of the numerical experiments, showing that our matheuristic performs very well and can be used to solve large-sized instances. Finally, Section 7 concludes the paper with some perspectives for future research.

2 Literature Review

The origin of IRP goes back to the paper of Bell et al. (1983) where the goal was to automatise the process of delivering liquid gases for the company Air Product. Since then, the interest for the IRP among scholars spiked and much work has been dedicated to present new variants of the IRP, related to its different parameters. Due to the rich literature on IRPs, scholars such as Moin and Salhi (2007), Andersson et al. (2010), Coelho et al. (2014) and Cao et al. (2020) review the literature and propose future perspectives in the area. Others, such as Malladi and Sowlati (2018) and Soysal et al. (2019), focused on the sustainability aspect of the IRP, whereas Roldán et al. (2017) review the literature of stochastic approaches. Bertazzi and Speranza (2012) focus on the solving approaches and propose an overview of the heuristics and matheuristics used for the IRP in the literature. As we do not aim to provide an exhaustive review of the IRP literature, this paper proposes instead a review of IRP works where the travelling time is the main focus.

2.1 Inventory Routing Problem with travelling time focus

In the IRP literature, the most common variants where the travelling time is the main focus is the IRP with time windows. In this case, clients can only be served within a certain time interval. Delgado et al. (2018) propose a review of the literature for the IRP with time windows. In a recent work by Ortega et al. (2020), the authors propose a matheuristic to solve the Consistent IRP with Time Windows and Split Deliveries. This variant of the IRP arises from a real-world application, namely route planning and inventory management for beer and other beverages companies. As the clients have different opening times and time windows, satisfying the overall demand can only be done by splitting the deliveries across more than one vehicle. Furthermore, the satisfaction of the clients depends on the consistency in delivery times. The matheuristic builds an initial solution by using a constructive heuristic to decide which clients must be visited for each period. Local search operators are then applied on this solution to improve its quality, and then a MIP is solved to determine the timing and the quantities. This solution goes through iterative improvements in a second phase, using an adaptive large neighbourhood search algorithm. Most recently, Wu et al. (2021) extend the problem to the location IRP with time windows and fuel consumption. The authors propose a two-stage hybrid metaheuristic algorithm to address this problem where in the first stage a customised genetic algorithm is solved and in the second, a gradient descent algorithm is proposed.

Another variant in the literature is the travel time-constrained IRP. In this context, the tours must be completed before a certain duration limit. This variant caters for some legislation requirements where drivers are not allowed to drive for longer than a certain duration in order to avoid traffic accidents. Lefever et al. (2021) present a Bender's decomposition algorithm to propose robust solutions where the travelling times evolve in symmetric and bounded intervals around their mean values. Another work is proposed by Coelho et al. (2020) in a Multi-Attribute Inventory Routing Problem context, which is the integration of the Multi-Depot IRP and the travel-time constrained IRP. In this paper, the authors propose a hybrid exact algorithm to solve the problem combining Mixed Integer Programming (MIP) and Variable Neighbourhood Search schemes. Extensive experimental results prove the efficiency of the hybridisation process, as it accelerates the resolution with respect to a branch-and-cut algorithm applied to the regular MIP formulation.

Other scholars focus on the sustainability of the transportation component of the IRP. In Alkaabneh et al. (2020), the authors propose a mathematical formulation that optimises the costs due to fuel consumption, inventory holding, and greenhouse gas emissions. Greenhouse gas emissions are computed as a function of fuel consumption levels that are calculated from the vehicle speed, load and travelled distance. As the travelling time is computed by vehicle speed and travelled distance, this paper is relevant for our literature. To solve the problem efficiently, the authors propose a Bender's decomposition approach with several acceleration strategies such as valid inequalities and efficient upper bounds.

In Li et al. (2014), an original way of handling travelling time in the IRP is proposed. Instead of optimising the classic inventory and transportation costs, the authors minimise the maximum travel time among all vehicles. The problem is set for a large petroleum and petrochemical Chinese enterprise group that is responsible for the distribution of gasoline to gas stations. The authors argue that in this context, avoiding stock out is more important than focusing on transportation cost minimisation, as running out of stock might not only be viewed as a business problem, but also a social problem by the local community. The authors propose a tabu-search algorithm to solve the problem, and propose a Lagrangian relaxation formulation to produce tight lower bounds in order to assess the efficiency of their algorithm.

In all the works cited above, the travelling time is an important parameter of the problem.

Most of these works model the travelling time between two locations as a constant value, which does not take into account the volatility of the travelling time, especially in urban logistics where traffic conditions can have a huge impact. However, in works such as Rahimi et al. (2017); Dong et al. (2018); Lefever et al. (2021), it is taken into consideration by modelling the travelling time as uncertain data. One other way to take this volatility into consideration is to consider time-dependent travelling times.

To the best of our knowledge, only one paper tackles the TD-IRP. In Cho et al. (2014), the authors propose a variant of the IRP where the speed of the vehicles is time-dependent. In order to model the volatility of the speed throughout the day, an artificial benchmark is generated where the day is decomposed into three main time intervals: morning rush hours, off-peak hours and evening rush hours. The speed is different from one interval to another and the authors assume that these assumptions are sufficient to mimic the traffic conditions during the day. Moreover, the authors propose a genetic algorithm to solve the problem. However, since this paper, no scholars have taken interest in the TD-IRP, thus we turn to pure routing problems to better understand how time-dependent travelling times are handled.

2.2 Time-dependent routing problems

In time-dependent routing problems, the travelling time between two locations does not only depend on the departure and arrival locations but also on the time of departure. Gendreau et al. (2015) propose an extensive review of the literature. They show that the time-dependent aspect is only considered for pure routing problems such as the Time-Dependent Travelling Salesman Problem (TD-TSP) or the Time-Dependent Vehicle Routing Problem (TD-VRP). The paper concludes that time-dependent problems are harder to solve than their basic counterparts. Moreover, although the literature is substantial, it is still quite recent and the resolution aspect can be further investigated, thus the need for new efficient approaches. The remainder of this section focuses only on work published subsequently to the review by Gendreau et al. (2015).

As time-dependent routing problems are hard to solve, most of the recent work on the TD-VRP and TD-TSP consists in proposing new efficient algorithms for their different variants. New exact algorithms are proposed to solve time-dependent routing problems using the most common approaches such as integer linear programming (Montero et al., 2017; Hansknecht et al., 2018; Arigliano et al., 2019), dynamic programming (Lera-Romero et al., 2020) and constraint programming (Melgarejo et al., 2015). The authors of Minh Vu et al. (2018) propose a novel approach based on the dynamic discretisation discovery framework that, instead of generating a time-expanded network in a static fashion, does so in a dynamic and iterative manner. The results show that the algorithm outperforms those of the literature and that it is robust with respect to all instance parameters, particularly the degree of travel time variability. However, although the performances of exact approaches are rapidly increasing, solving large-sized problems is still a computational challenge. Therefore, scholars propose algorithms based on local search procedures to solve such instances, such as adaptive large neighbourhood search (Franceschetti et al., 2017; Rincon-Garcia et al., 2020; Pan et al., 2020), ant colony algorithm (Deng et al., 2018; Liu et al., 2020), tabu search (Ban, 2019), variable neighbourhood search (Lu et al., 2020), or genetic algorithms such as NSGA-II (Zhao et al., 2019).

Another part of the time-dependent literature focuses on generating time-dependent travelling time functions. A recent work in this area is proposed by Rifki et al. (2020), where the authors propose a new real-life benchmark for routing problems based on the traffic conditions of the city of Lyon in France, using a dynamic microscopic simulator of traffic flow. The purpose of their study is to show the impact of space granularity, i.e. the number of sensors deployed to monitor the traffic flow, and time granularity, i.e. the number and length of time steps, on the quality of the solutions for pick-up and delivery optimal tours. Other scholars take the volatility of travelling time functions further by expanding it to an uncertain time-dependent environment using fuzzy logic to model such uncertainties (Gambuzza et al., 2018; Koczy et al., 2019; Almahasneh, 2020).

This rich time-dependent routing literature provides benchmarks to experiment on and ideas on how to model and solve the TD-IRP that will be exploited throughout this paper.

3 Problem description and modelling

The IRP is set in a network where a supplier is responsible for managing the inventory and organising the replenishment of a set of clients over a time horizon. The supplier has a production rate whereas the clients have a demand to satisfy, a capacitated inventory and a service time in which the products are unloaded. The objective is to decide, for each period of the time horizon, the replenishment quantities as well as planning routing sequences, in order for the clients to satisfy their demand, while minimising the total costs incurred by both inventory holding and transportation. The TD-IRP is an extension of the IRP on its transportation component, where the travelling time between two locations does not only depend on the departure and arrival locations, but on the time of departure as well. In this paper, we consider that a period represents a day of the week. Each day is discretised into small time intervals for which the travelling time between two locations are different from one time interval to another, and are constant within one time interval. These time intervals are called time steps. Moreover, in this paper, we consider that the cost of transport incurs only when the vehicle is moving. Therefore, a tour can contain waiting times, as it may sometimes be more efficient to leave later to improve the cost of the tour. However, in order for the waiting times to not be exaggerated, a travel-time constraint is imposed on the duration of the tours. Finally, we consider that all tours start at the beginning of the period. This assumption may seem counterproductive in the case of TD-IRP, since leaving later may reduce the cost of the tour. Note however that in real-life situations, the working hours of the drivers are generally fixed beforehand. In such a case, starting at a different time than the beginning of the period would amount to adding a waiting time at the depot.

3.1 Mathematical formulation

Let $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ be a graph where vertex $0 \in \mathcal{V}$ represents the supplier, $\mathcal{V}' = \mathcal{V} \backslash \{0\}$ is the set of clients and $\mathcal{A}$ is a set of arcs linking them. $\mathcal{H} = \{1, 2, ..., |\mathcal{H}|\}$ is the scheduling time horizon and $\mathcal{H}' = \{0\} \cup \mathcal{H}$ is the horizon including period 0 which represents the initial state. $p \in \mathcal{H}'$ represents the index of the period. $\mathcal{M}$ is a set of time steps where $m \in \mathcal{M}$ represents the index of the time step and L its duration. $|\mathcal{M}|$ represents the number of time intervals into which a period $p \in \mathcal{H}$ is discretised. Finally, $\mathcal{M}'$ is the subset of time steps which satisfy the travel-time constraint, i.e. all tours must be completed before the end of time step $|\mathcal{M}'|$.

Each client $i \in \mathcal{V}'$ has a demand D_i^p for period $p \in \mathcal{H}$, an initial inventory I_i^0, a maximal inventory level $I_i^{\max}$ and a service time s_i. The supplier has an unlimited inventory capacity, an initial inventory I_0^0, R^p products available at each period $p \in \mathcal{H}$ and a vehicle with a maximal capacity C. Keeping one item in inventory for a period incurs a holding cost h_i for each actor $i \in \mathcal{V}$. Finally $f(i, j, m)$ is a travelling time function that represents the duration of travelling through arc $(i, j) \in \mathcal{A}$ when leaving $i \in \mathcal{V}$ at time step m and c is the cost of one unit of time travelled.

Variables: let x_{ij}^p be a binary variable that equals 1 if arc $(i, j) \in \mathcal{A}$ is travelled in period p, 0 otherwise. x_{ij}^{pm} is a binary variable that is equal to 1 if $(i, j) \in \mathcal{A}$ is travelled in period p and the departure from i to j is in time step m, 0 otherwise. It is the same variable as x_{ij}^p, only with a different granularity since if $|\mathcal{M}| = 1$, $x_{ij}^p = x_{ij}^{pm}$. y_i^p is a binary variable that is equal to 1 if location $i \in \mathcal{V}$ is visited in period p, 0 otherwise. $I_i^p \in \mathbb{R}$ represents the inventory level of actor $i \in \mathcal{V}$ at the end of period $p \in \mathcal{H}'$ and $q_i^p \in \mathbb{R}$ is the quantity sent from the supplier to client $i \in \mathcal{V}'$.

Model: The mathematical model (TD-IRP) is a generic model inspired by the one presented in Archetti et al. (2007) which is widely used in the IRP literature. The objective computes the total holding cost for each location $i \in \mathcal{V}$ and time period $p \in \mathcal{H}'$ and the total travelling cost for all time periods $p \in \mathcal{H}$. Constraints (1) are flow conservation constraints that compute the inventory level of the supplier at each period $p \in \mathcal{H}$ from its previous inventory level, the quantity produced at p and the quantities sent to the clients at p. Similarly, constraints (2) state the flow conservation constraints regarding the clients. They compute the inventory level of each client $i \in \mathcal{V}'$ for each period $p \in \mathcal{H}$ from its previous inventory level, the quantity received from the supplier and its demand for period p. The inventory capacity is enforced through several sets of constraints: Constraints (3) state that the inventory level of client $i \in \mathcal{V}'$ at any period $p \in \mathcal{H}$ must be lower than $I_i^{\max}$, and constraints (4) state that a replenishment of this

TD-IRP					
min	$obj = c \sum_{(i,j)\in\mathcal{A}} \sum_{p\in\mathcal{H}} \sum_{m\in\mathcal{M}} f(i,j,m) \times x_{ij}^{pm} + \sum_{i\in\mathcal{V}} \sum_{p\in\mathcal{H}'} h_i \times I_i^p$				
s.t	I_0^p	$=$	$I_0^{p-1} - \sum_{i\in\mathcal{V}'} q_i^p + R^p$	$\forall p \in \mathcal{H}$	(1)
	I_i^p	$=$	$I_i^{p-1} + q_i^p - D_i^p$	$\forall i \in \mathcal{V}'$, $\forall p \in \mathcal{H}$	(2)
	I_i^p	$\leq$	$I_i^{\max}$	$\forall i \in \mathcal{V}'$, $\forall p \in \mathcal{H}$	(3)
	$q_i^p + I_i^{p-1}$	$\leq$	$I_i^{\max}$	$\forall i \in \mathcal{V}'$, $\forall p \in \mathcal{H}$	(4)
	q_i^p	$\leq$	$y_i^p \times I_i^{\max}$	$\forall i \in \mathcal{V}'$, $\forall p \in \mathcal{H}$	(5)
	q_0^p	$\leq$	$C \times y_0^p$	$\forall p \in \mathcal{H}$	(6)
	$\sum_{j\in\mathcal{V}'} x_{ij}^p$	$=$	y_i^p	$\forall i \in \mathcal{V}$, $\forall p \in \mathcal{H}$	(7)
	$\sum_{j\in\mathcal{V}'} x_{ji}^p$	$=$	y_i^p	$\forall i \in \mathcal{V}$, $\forall p \in \mathcal{H}$	(8)
	$\sum_{(i,j)\in\mathscr{S}} x_{ij}^p$	$\leq$	$\lvert\mathscr{S}\rvert - 1$	$\forall \mathscr{S} \subseteq \mathcal{A}$, $p \in \mathcal{H}$	(9)
	$\sum_{m\in\mathcal{M}} x_{ij}^{pm}$	$=$	x_{ij}^p	$\forall (i,j) \in \mathcal{A}$, $\forall p \in \mathcal{H}$	(10)
	$\sum_{j\in\mathcal{V}'} x_{0j}^{p0}$	$=$	y_0^p	$\forall p \in \mathcal{H}$	(11)
	$\sum_{v_k\in P\setminus\{v_n\}} \sum_{m_k\in T} x_{v_k,v_{k+1}}^{p,m_k}$	$\leq$	$\lvert P\rvert - 2$	$\forall [P,T]$ infeasible, $p \in \mathcal{H}$	(12)
	y_i^p	$\in$	$\{0,1\}$	$\forall i \in \mathcal{V}$, $\forall p \in \mathcal{H}$	(13)
	q_i^p	$\geq$	0	$\forall i \in \mathcal{V}'$, $\forall p \in \mathcal{H}$	(14)
	I_i^p	$\geq$	0	$\forall i \in \mathcal{V}$, $\forall p \in \mathcal{H}$	(15)
	x_{ij}^p	$\in$	$\{0,1\}$	$\forall (i,j) \in \mathcal{A}$, $\forall p \in \mathcal{H}$	(16)
	x_{ij}^{pm}	$\in$	$\{0,1\}$	$\forall (i,j) \in \mathcal{A}$, $m \in \mathcal{M}'$, $\forall p \in \mathcal{H}$	(17)

client at period $p \in \mathcal{H}$ cannot exceed its maximal inventory level. Constraints (5) link variables y_i^p with q_i^p, stating that a client $i \in \mathcal{V}'$ who receives a quantity at period $p \in \mathcal{H}$, is necessarily visited at p. $I_i^{\max}$ is used here as an upper bound for q_i^p. Constraints (6) work similarly for the supplier, stating that the quantity leaving supplier 0 at period $p \in \mathcal{H}$ is limited by the vehicle capacity C. Constraints (7) and constraints (8) are flow conservation constraints for the routing component for each $i \in \mathcal{V}'$ and respectively state that if a client is visited, one arc arrives to it and another leaves from it. Constraints (9) eliminate sub-tours, where $\mathscr{S}$ is a set of sub-tours. Constraints (13) to (16) enforce integrality and non-negativity conditions on the IRP variables.

The time-dependent aspect of the IRP is ensured by constraints (10) to (12) and (17). Constraints (10) link variables x_{ij}^{pm} with variables x_{ij}^p and state that if an arc $(i,j) \in \mathcal{A}$ is travelled in period p, it leaves i at one and only one time step $m \in \mathcal{M}$. Constraints (11) state that routing starts at the beginning of period $p \in \mathcal{H}$, i.e. $m = 0$. Constraints (12) eliminate time-dependent infeasible paths, where $[P, T]$ is a time-dependent infeasible path (see definition below). Finally, constraints (17) enforce integrality and non-negativity conditions on the TD-IRP variables.

Constraints (9) and Constraints (12) are exponential constraints. However, only a part of them will be dynamically generated in a branch-and-cut procedure that will be described in subsection 5.1. There are various strategies to express the sub-tour constraint in a MILP (Orman and Williams, 2006), however none would yield a model that could be solved in a reasonable amount of time by a MILP solver.

The definition of a time-dependent infeasible path, with allowed waiting times, is presented below, as described in Miranda-Bront et al. (2010).

Time-dependent infeasible path

Let $P =< v_1, ..., v_{k-1}, v_k, v_{k+1}, ..., v_n >$ where $v_k \in \mathcal{V}'$ and $v_1 = v_n = 0$ the supplier. Let $T =< m_1, ..., m_{k-1}, m_k, m_{k+1}, ..., v_{n-1} >$ be a set of departure time steps. A time-dependent path $[P, T]$ is a combination of P and T where T represents the departure time steps of $v_k \in P \backslash \{v_n\}$.

Let t_{v_k} be the earliest departure time, s_{v_k} the service time at location $v_k \in P$ and $t^{\min}_{m_k}$ the beginning of time step m_k:

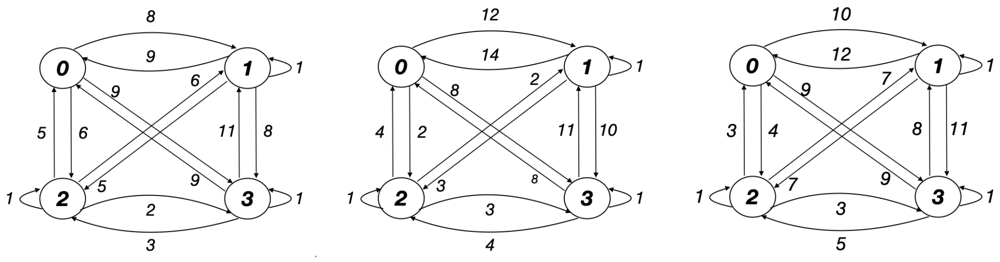
- $t_{v_k} = \begin{cases} 0 + s_{v_k} & v_k = 0 \\ \max\{t_{v_{k-1}} + f(v_{k-1}, v_k, \lfloor \frac{t_{v_{k-1}}}{L} \rfloor) + s_{v_k},\ t^{\min}_{m_k}\} & \forall k \in P \backslash \{v_n, v_1\} \end{cases}$
- $[P, T]$ is infeasible $\iff \exists v_k \in P : t_{v_k} \notin [t^{\min}_{m_k}; t^{\min}_{m_k+1}[$

3.2 An illustrative example

Let us consider an example of the TD-IRP where the network is composed of a supplier and three clients $\mathcal{V} = \{0, 1, 2, 3\}$, a time horizon $|\mathcal{H}| = 3$ and a vehicle capacity $C = 20$. Each period $p \in \mathcal{H}$ is decomposed into $|\mathcal{M}| = 3$ time steps with a length $L = 10$ time units. Table 1 and Figure 1 present all the data related to instance w. The columns of Table 1 represent, respectively, the indices i of the supplier/clients, their initial inventory I^0_i and maximum inventory $I^{\max}_i$, the production rate R^p of the supplier, the demand D^p_i of the clients and finally the holding costs. Figure 1 illustrates the travelling times between each couple of locations for every time step. Arcs going from client i to i represent the service time at the client. An example of the form of the travelling time function f between the supplier 0 and client 1 is presented in Figure 2. The figure shows that the travelling time between 0 and 1 is of $f(0, 1, 1) = 8$ for the first time step, 12 for the second and 10 for the third.

i	I_i^0	$I_i^{\max}$	R^p			D_i^p			h_i
			$p=1$	$p=2$	$p=3$	$p=1$	$p=2$	$p=3$	
0	30	$+\infty$	30	30	30	-	-	-	0.1
1	15	20	-	-	-	8	12	10	0.15
2	10	25	-	-	-	15	11	19	0.18
3	5	10	-	-	-	8	4	3	0.13

Table 1: A representation of the data for the example



(a) $m = 1$ (b) $m = 2$ (c) $m = 3$

Figure 1: Time-dependent graphs for each time step of the example

A solution for the example is represented in Table 2 and Figure 3. Table 2 presents the inventory levels at the end of each period of each location I_i^p and the quantities q_i^p sent from the supplier to each client i for the whole time horizon. Figure 3 shows the sequence of the routing component through a Gantt chart.

As we can see from Table 2, clients 2 and 3 are replenished in period $p = 1$, all clients are replenished in period $p = 2$ and clients 1 and 2 are visited in period $p = 3$. Figure 3 shows that in period $p = 1$, the vehicle leaves the supplier, visits client 2 and then client 3 in the first time step. In the beginning of the second time step, it leaves client 3 to return to the supplier without waiting at any node. In period $p = 3$ on the other hand, the vehicle visits client 1 in the first time step. It waits until the beginning of the second

i	I_i^p			q_i^p		
	$p=1$	$p=2$	$p=3$	$p=1$	$p=2$	$p=3$
0	40	50	60			
1	7	8	0	0	13	2
2	10	1	0	15	2	18
3	2	3	0	5	5	0

Table 2: Values of variables I_i^p and q_i^p for a solution of the example

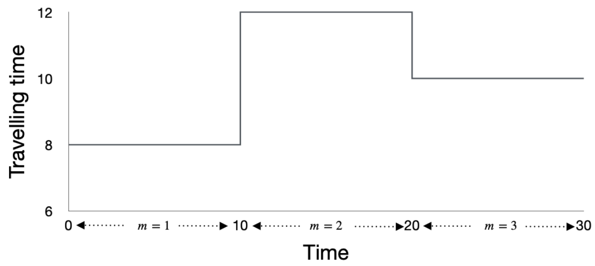


Figure 2: A representation of the travelling time function between the supplier 0 and client 1.

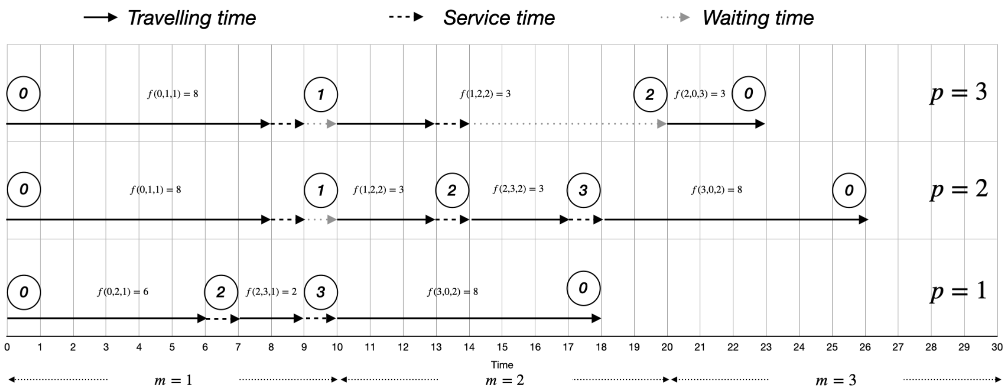


Figure 3: A representation of the transportation component for a solution of the example

time step to visit client 2 and then waits for the third time step to get back to the supplier.

4 A Time-Dependent Inventory Routing benchmark

Since the TD-IRP literature is rather sparse, benchmarks for the TD-IRP are almost nonexistent. In this section, we thus propose a new benchmark set for the TD-IRP, combining an IRP benchmark proposed in Archetti et al. (2007) with a TD-TSP benchmark proposed in Rifki et al. (2020). Both benchmarks are first modified and adapted to better fit our needs, the main objective being of mimicking real-life conditions as closely as possible.

4.1 Inventory management data

The inventory-related data of the benchmark is generated using the benchmark proposed in Archetti et al. (2007). Although this benchmark is the most commonly used in the IRP literature, it is generated following rather strong assumptions:

- the demand of the clients is constant for the whole horizon;
- the vehicle's capacity is large enough to replenish all clients in one period;
- initial inventories are such that no replenishment is needed for the first period of the horizon;
- the holding cost of some clients is inferior than the one of the supplier.

These assumptions are rarely met in real-life instances. Furthermore, they lead to optimal solutions in which the clients are all replenished in a single period. Such structures rarely reflect real-life solutions, especially in urban logistics where the size of vehicles is small and the inventory capacities at the delivery points are not large enough to ensure demand satisfaction for a large period of time. Moreover, if all the clients are replenished in a single period over the time horizon, the problem is almost no longer an IRP but rather a TSP or a VRP.

Therefore, modifications are performed on a subset of the instances proposed, in order to provide a more realistic benchmark. The generation of the new values is done for instances with $|\mathcal{H}| = \{3, 6\}$, a number of clients $|\mathcal{V}'_{|\mathcal{H}|=3}| = \{5, 10, 15, 20, 25, 30\}$ when $|\mathcal{H}| = 3$ and $|\mathcal{V}'_{|\mathcal{H}|=6}| = \{5, 10, 15, 20\}$ when $|\mathcal{H}| = 6$. For each combination of these parameters, 5 different instances are generated, which yields a total of 50 instances. Algorithm 1 presents the generation process.

The demand of the new benchmark is no longer constant, but varies from one period to another by $\pm[-0.25, 0.25]$ of the initial value D_i^{old}. The value of 0.25 in this case is arbitrary. The production rate of the supplier is set such that it is possible to meet the demand of all the clients for each period. The inventory capacity of the clients can cover the demand for up to two periods on average whereas the one of the supplier is unlimited. The initial inventory levels of the clients follow a uniform distribution over the set $\{0, 0.25, 0.50, 0.75, 1\} \times I_i^{\max}$. The initial inventory of the supplier can cover the inventory capacity of all the clients once. The vehicle capacity is set such that it can only cover up to 90% of the demand average for one period. Finally, we assume that the holding cost of the supplier should be inferior to the one of the clients and set it to 2/3 of the smallest holding cost of the clients. This assumption is made from the realistic hypothesis that the supplier is able to handle its inventory more efficiently than its clients. Furthermore, clients in this case are situated in urban areas, where the holding costs tend to be higher.

Algorithm 1: Inventory management component generation

input: an instance from the benchmark of Archetti et al. (2007)
for $i \in \mathcal{V}'$ **do**
  for $p \in \mathcal{H}$ **do**
    generate a random $\delta \in [-0.25, 0.25]$
    $D_i^p = \lfloor D_i^{\text{old}} \times (1+\delta) \rceil$
  end for
  $I_i^{\max} = \frac{2}{|\mathcal{H}|} \sum\limits_{p \in \mathcal{H}} D_i^p$
  generate a random number $\rho \in [0, 1[$
  $$I_i^0 = \begin{cases} 0 & \text{if } \rho \in [0, 0.2[\\ I_i^{\max} \times 0.25 & \text{if } \rho \in [0.2, 0.4[\\ I_i^{\max} \times 0.5 & \text{if } \rho \in [0.4, 0.6[\\ I_i^{\max} \times 0.75 & \text{if } \rho \in [0.6, 0.8[\\ I_i^{\max} & \text{if } \rho \in [0.8, 1[\end{cases}$$
end for
$C = \frac{0.9}{|\mathcal{H}| \times |\mathcal{V}'|} \sum\limits_{i \in \mathcal{V}'} \sum\limits_{p \in \mathcal{H}} D_i^p$
$h_0 = \frac{2}{3} \min\limits_{i \in \mathcal{V}'} h_i$
$R^p = \frac{1.25}{|\mathcal{H}| \times |\mathcal{V}'|} \sum\limits_{i \in \mathcal{V}'} \sum\limits_{p \in \mathcal{H}} D_i^p$
$I_0^0 = \sum\limits_{i \in \mathcal{V}'} I_i^{\max}$

4.2 Time-dependent travelling time functions

In the benchmark of Archetti et al. (2007), the distance travelled between two locations is defined by the euclidean distance. This data is disregarded and replaced by time-dependent travelling time functions.

Producing time-dependent functions for routing problems is a productive field in transportation literature. A variety of functions exist in the literature (Rifki et al., 2020): some are artificial while others are based on real traffic data. Rifki et al. (2020) propose a benchmark based on the traffic conditions of the city of Lyon in France, using a dynamic microscopic simulator of traffic flow. Based on data collected from sensors placed in the axes of the city, a time-dependent travelling time function is defined for a time interval of 12 hours and is decomposed into time steps. For each time step, a consistent spatio-temporal mean formulation is used to compute the travelling time for each segment of the network. Afterwards, the shortest path is computed between each two different locations for each time step. The benchmark yields a set of constant piece-wise travelling time functions between each two different locations with different time granularity/time steps $|\mathcal{M}| = \{1, 12, 30, 60, 120\}$ of respective lengths $L = \{720, 60, 24, 12, 6\}$ minutes, for an instances total of 250. An example of these travelling time functions between two random locations is presented in Figure 4. It shows the travelling time function, as explained in Figure 2, but with different time steps $|\mathcal{M}|$. The figure shows that the travelling time functions are highly volatile which can seem unrealistic. However, in the case of urban transportation, the travelling times is indeed volatile, because traffic congestion can have a big impact on travelling times, especially small ones that can double or triple.

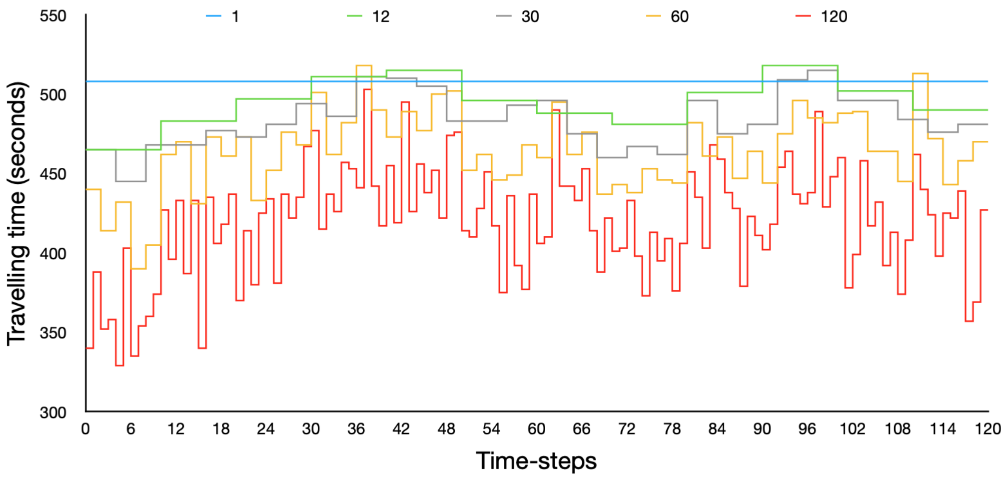


Figure 4: A travelling time function between two random locations for $|\mathcal{M}| = \{1, 12, 30, 60, 120\}$

For all instances described in section 4.1, a set of locations are randomly picked from Rifki et al. (2020)'s benchmark and matched to each location from Archetti et al. (2007)'s benchmark. The time-dependent travelling time functions corresponding to this set of locations are added to the instance.

4.3 First In First Out (FIFO) property

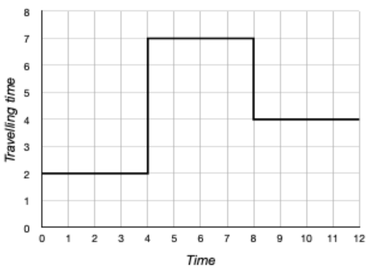
The problem of such constant piece-wise functions is that they do not necessarily satisfy the FIFO property.

Definition: Let $[t_m^{\min}, t_{m+1}^{\min}[$ be the time interval of time step m, $t_m^{\min}$ is the beginning of time step m. A travelling time function f that satisfies the FIFO property is such that:

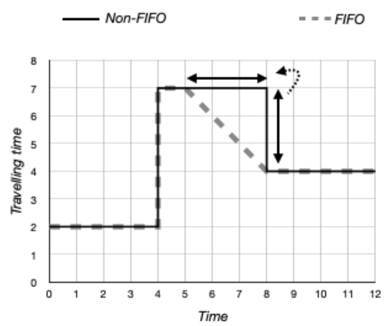
$t' + f(i, j, m') \geq t + f(i, j, m) \quad \forall (i, j) \in \mathcal{A}, \forall t \in [t_m^{\min}, t_{m+1}^{\min}[, \forall t' \in [t_{m'}^{\min}, t_{m'+1}^{\min}[$ where $m' \geq$

m

In other words, when leaving i to j at t', the arrival time must be greater than the one when leaving i to j at t, where $t' \geq t$. This property is violated for every decreasing discontinuity in f, i.e. for all $m \in \mathcal{M}$ where $f(i,j,m) > f(i,j,m+1)$. However, since $f(i,j,m)$ represents, by definition, the shortest path possible between i and j when leaving at m, and since, in urban transportation, the travelling speed is approximately constant, the FIFO property must be at all times satisfied for our benchmark. One way to ensure that the FIFO property is enforced when handling constant step-wise travelling time functions is to transform such functions into linear step-wise functions that we will call f^{FIFO}. Ichoua et al. (2003) apply this procedure for functions that represent the speed between two locations. However, since we do not have time-dependent travel speeds at our disposal but travelling times instead, our transformation is based on the work of Fleischmann et al. (2004). This transformation smooths the function by mimicking waiting at nodes for the next time step. Figure 5 shows how the transformation from a constant step-wise function to a linear one is made. For a more detailed description of the transformation algorithm, please refer to Touzout et al. (2020).



(a) constant step-wise function



(b) linear step-wise function

Figure 5: Transformation of a constant step-wise to a linear step-wise function

It is worth noting that our transformation is made only for time steps where a decreasing discontinuity happens just as in Melgarejo et al. (2015), as opposed to Fleischmann et al. (2004), where the function is smoothed for each increasing or decreasing discontinuity. This choice was made to ensure that the FIFO property is enforced, but at the same time stay as faithful as possible to the benchmark. Moreover, we believe that this makes the travelling time functions more realistic as when a congestion appears, the travelling time does not evolve in a smooth manner but evolves as sharply as shown in Figure 4.

A problem that arises when using f^{FIFO} is that in order to compute the cost of travelling between two locations i and j, knowing the time step of departure is not enough. Instead, the exact second of the departure is needed since f^{FIFO} is linear step-wise. This means that in order to use such a function, the length of a period needs to be discretised into very small granularities, i.e. of the order of the seconds, which yields models with an extremely high number of variables. Another way is to formulate the problem in another manner. Sun et al. (2018) propose a flow formulation referred to in the literature as the "time-dependent ready function formulation". This formulation is proposed for the TD-TSP with time windows. However, the formulation is based on big-M constraints. As we do not tackle the TD-IRP with time windows, such constraints are known to be highly inefficient for problems where the M cannot be well-restricted, which is the case of this paper. Such models are unsolvable even for the smallest instances. Therefore, in this paper we enforce the FIFO property by using a constant piece-wise function and allowing waiting at nodes. This procedure emulates the transformation algorithm.

4.4 Travel-time constraint

In section 3.1, we state that travelling cost incurs only when the vehicle is moving from one location to another. Therefore, since waiting at nodes does not incur any cost, it is unrestricted and can occur as long as it is profitable to wait. However, this can yield solutions where the waiting time is extremely high in comparison to the travelling time. This is unrealistic as one cannot ask a driver to park and wait for hours in order to optimise the cost. In order to restrict this waiting time, we propose to limit the total duration of a tour with travel time constraints.

Since our benchmark is set in an urban distribution context, and since a day is long enough to visit a high number of clients in one period, we propose travel time constraints that allow the visit of all clients in one period as long as the capacity of the vehicle is not exceeded, while ensuring the FIFO property and at the same time restrict the waiting times. To generate such values, a TD-TSP is solved through an iterated local search heuristic presented in Algorithm 2, using f^{FIFO}. The objective value of the TD-TSP will be the travel-time constraint for which the tour must be completed, thus minimising the waiting at nodes. Note that any heuristic that can solve the TD-TSP can provide a value for $\mathcal{M}'$.

The obtained benchmark set is readily available at `https://github.com/faycalt/TDIRP`.

5 Investigating the relevance of TD-IRP

In this section, we investigate the relevance of using time-dependent travelling time functions instead of constant ones on the objective function as well as the structure of optimal solutions. In order to do so, we compare optimal time-dependent solutions to constant travelling time solutions re-solved in a time-dependent environment. To that purpose, all time-dependent instances, i.e. with $|\mathcal{M}| > 1$, are solved twice. The two phases are described in Figure 6. In the first phase, the instances are solved optimally. Afterwards, the solutions of instances with constant travelling time, i.e. with $|\mathcal{M}| = 1$,

Algorithm 2: TD-TSP: ILS

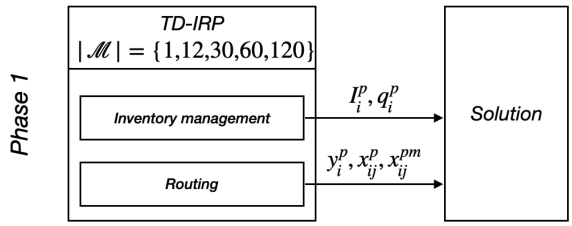
input: f^{FIFO}, a set of locations to visit and an empty solution $\mathcal{S}^{\text{TDTSP}}$
while A time limit is not exceeded and the local optimal is not reached **do**
  generate a random sequence and set it to $\mathcal{S}^{\text{TDTSP}}$ if it is better or is still empty
  apply local search operations such as movements and swaps
  Movements:
  for $k, k' \in V, k \neq k'$ **do**
    slide the vertex at the k^{th} position and slide it to position k' leaving the rest of the sequence unchanged
  end for
  Swaps:
  for $k, k' \in V, k \neq k'$ **do**
    swap the vertex at the k^{th} position with the vertex of position k' and vice versa
  end for
  set $\mathcal{S}^{\text{TDTSP}}$ to the obtained solution if it is better
end while
$|\mathcal{M}'| = \lceil \frac{S^{TDTSP}_{\text{objective}}}{L} \rceil$
return $\mathcal{M}'$

are re-solved in time-dependent instances, where $|\mathcal{M}| \in \{12, 30, 60, 120\}$. As we can see from Figure 6b, by solving the problem with constant travelling time, we fix the inventory management variables I_i^p, q_i^p, which state the inventory levels and the quantities sent for each client, and the routing variables x_{ij}^p which state the clients to be visited and in which order they should be visited. Afterwards, the time steps of departure from each location are determined subsequently by solving the TD-IRP in a time-dependent environment.

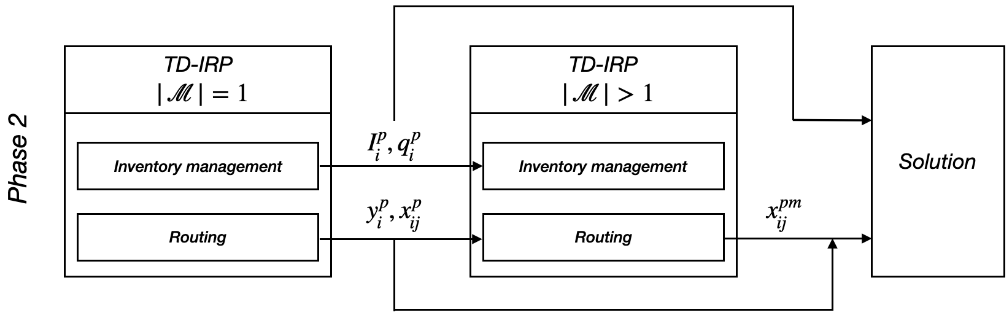
5.1 A Branch & Cut procedure

In order to solve the different TD-IRPs optimally, a branch-and-cut algorithm is used. The mathematical formulation presented in Section 3.1 is strengthened with additional valid inequalities presented in Archetti et al. (2007), Coelho and Laporte (2014) and Desaulniers et al. (2016). Archetti's valid inequalities are "logical inequalities" and are inspired by logical cuts introduced for problems such as the Orienteering Problem and the Undirected Selective TSP. Coelho's valid inequalities determine the minimum number of routes in the planning horizon, whereas Desaulniers's valid inequalities determine "the minimum number of sub-deliveries per demand". Moreover, the bounds are improved and the routing component of the IRP re-formulated according to the work of Lefever (2018). Finally, sub-tour and time-dependent infeasible path elimination constraints are added dynamically into the procedure as described in Algorithm 3.

It is worth noting that the contribution in relation to the solving approach lies on



(a) Procedure for optimal TD-IRP solutions



(b) Procedure for constant travelling time solutions re-solved in a time-dependent environment

Figure 6: Comparison procedure

Algorithm 3: Branch-and-cut procedure

input: an empty set of nodes $\mathcal{Q}$ and an empty solution $\mathcal{S}$
add the root node to $\mathcal{Q}$ and set the upper bound $\bar{b} = +\infty$
while $\mathcal{Q}$ is not empty **do**
  solve the linear relaxation of the first node
  if the solution of the linear relaxation is fractional **then**
    branch on fractional variables, add the resulting nodes to $\mathcal{Q}$ and delete the first node
  else
    if subtours $\mathscr{S}$ exist **then**
      add corresponding constraints (9)
    else
      if time-dependent infeasible paths $[P, T]$ exist **then**
        add corresponding constraints (12)
      else
        update $\bar{b}$ and $\mathcal{S}$ if needed and delete the node from $\mathcal{Q}$
      end if
    end if
  end if
end while
return $\mathcal{S}$

combining cuts and reformulations from the literature of the IRP and extending them with the time-dependent infeasible paths elimination constraints (12).

5.2 Numerical experiments

All experiments are conducted on a CPU Intel Xeon E5-1620 v3 @3.5Ghz with 64GB RAM in a Java-Gurobi environment. Gurobi 9.0.2 is used as a solver and the dynamic constraints are added using the lazyConstraints parameter. The experiments are conducted on all 250 instances generated in Section 4 with a time limit of 3600 seconds.

The results are presented in table 3 as follows: Columns $|\mathcal{H}|$, $|\mathcal{V}'|$, and $|\mathcal{M}|$ present, respectively, the length of the horizon, the number of clients in the network and the number of time steps of the travelling time function. Columns z, g and CPU present, respectively, the objective value, the gap to the best lower bound found and the execution time of the solution of the TD-IRP. These values are average values over five instances of each combination of parameters $\mathcal{H}$, $\mathcal{V}'$ and $\mathcal{M}$. Columns $z_{|\mathcal{M}|=1}$, $g_{|\mathcal{M}|=1}$ and $CPU_{\mathcal{M}} = 1$ represent, respectively, the objective value, the gap to the best lower bound and the execution time, on average, of the TD-IRP when the solution with constant travelling time is re-solved in a time-dependent environment. Finally, column $g^{z}_{z_{|\mathcal{M}|=1}}$ represents the gap between z and $z_{|\mathcal{M}|=1}$ where $g^{z}_{z_{|\mathcal{M}|=1}} = \frac{z_{|\mathcal{M}|=1} - z}{z}$

Table 3: Results

$\lvert\mathcal{H}\rvert$	$\lvert\mathcal{V}'\rvert$	$\lvert\mathcal{M}\rvert$	z	$g\%$	CPU	$z_{\lvert\mathcal{M}\rvert=1}$	$g_{\lvert\mathcal{M}\rvert=1}\%$	$CPU_{\lvert\mathcal{M}\rvert=1}$	$g^{z}_{z_{\lvert\mathcal{M}\rvert=1}}\%$
3	5	1	3790.27	0.00	0.00				
		12	3387.27	0.00	0.00	3398.27	0.00	0.00	0.32
		30	3136.02	0.00	0.01	3136.47	0.00	0.00	0.02
		60	2916.07	0.00	0.02	2926.07	0.00	0.00	0.34
		120	2584.09	0.00	0.05	2611.27	0.00	0.00	1.25
	10	1	5544.34	0.00	0.04				
		12	4906.04	0.00	0.06	4927.14	0.00	0.00	0.44
		30	4579.25	0.00	0.55	4628.34	0.00	0.01	1.09
		60	4183.26	0.00	1.10	4237.14	0.00	0.02	1.25
		120	3752.34	0.00	45.97	3829.54	0.00	0.05	2.02
	15	1	6822.30	0.00	0.07				
		12	6169.50	0.00	0.32	6230.70	0.00	0.01	0.99
		30	5823.55	0.00	4.46	5963.70	0.00	0.04	2.45
		60	5346.01	0.00	70.24	5470.30	0.00	0.23	2.37
		120	4701.56	0.45	1610.18	4885.90	0.00	1.17	3.94
	20	1	8588.07	0.00	0.28				
		12	7699.65	0.00	4.87	7833.87	0.00	0.02	1.68
		30	7198.26	0.00	782.55	7375.67	0.00	0.13	2.43
		60	6622.56	0.75	2240.37	6900.47	0.00	2.78	4.18
		120	6025.68	4.47	3600.03	6285.47	0.00	2.02	4.42
	25	1	9246.05	0.00	0.78				
		12	8426.63	0.00	160.87	8536.65	0.00	0.04	1.32
		30	7864.11	0.80	2308.50	8025.05	0.00	0.41	2.05
		60	7378.17	4.16	3600.03	7547.05	0.00	26.22	2.35
		120	6807.12	9.92	3600.04	6870.65	0.00	430.99	0.94
	30	1	10611.99	0.00	0.73				
		12	9615.59	0.29	2397.94	9710.79	0.00	0.15	0.97
		30	9160.58	2.85	3600.02	9220.39	0.00	4.81	0.63
		60	8599.19	6.21	3600.03	8606.79	0.19	1232.12	0.10
		120	7761.99	8.21	3600.10	7820.39	0.24	1825.42	0.76
6	5	1	8385.85	0.00	0.04				
		12	7514.45	0.00	0.03	7520.65	0.00	0.00	0.09
		30	6941.13	0.00	0.07	6942.45	0.00	0.00	0.02
		60	6448.17	0.00	0.21	6460.05	0.00	0.00	0.19
		120	5740.21	0.00	0.26	5796.45	0.00	0.01	1.04
	10	1	11040.83	0.00	0.11				
		12	9856.17	0.00	0.27	9968.03	0.00	0.00	1.15
		30	9209.18	0.00	13.60	9344.03	0.00	0.01	1.52
		60	8496.00	0.00	47.21	8604.63	0.00	0.05	1.32
		120	7532.50	0.39	1074.80	7777.83	0.00	0.21	3.30
	15	1	13923.93	0.00	0.47				
		12	12564.94	0.00	3.05	12683.13	0.00	0.02	0.91
		30	11923.00	0.00	495.22	12134.93	0.00	0.09	1.82
		60	10912.59	0.95	2929.27	11158.33	0.00	1.53	2.24
		120	9760.72	4.45	3600.02	9980.93	0.00	3.79	2.25
	20	1	17726.17	0.00	2.16				
		12	15905.55	0.02	1143.01	16034.57	0.00	0.04	0.78
		30	14821.00	1.28	3600.01	15030.17	0.00	0.59	1.39
		60	13695.51	3.52	3600.03	13945.17	0.01	72.53	1.80
		120	12501.52	7.31	3600.04	12641.77	0.06	1535.07	1.11

5.3 Discussion

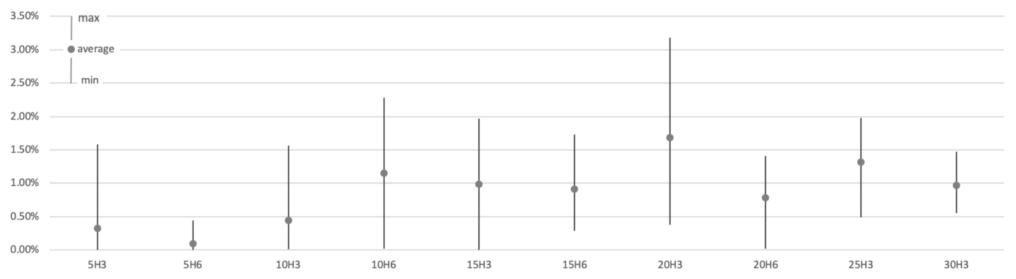
The results presented in Table 3 yield the following three observations:

Observation 1: *TD-IRPs are difficult to solve.* Indeed, instances with $|\mathcal{V}'| \geq 20$ and $|\mathcal{M}| \geq 30$, among others, are not optimally solved within the time limit. Although the TD-IRP is not strictly a routing problem, routing problems of the order of 20, 25 and 30 clients are generally considered, in the VRP and TSP literature, as small instances. However, in a time-dependent context, for an instance where $|\mathcal{M}| = 120$, all the routing variables are multiplied by $|\mathcal{M}|$ which makes the problem 120 times bigger. Furthermore, the periodicity aspect adds to the difficulty by multiplying all variables by $|\mathcal{H}|$. Finally, since the problem is an integrated one, inventory decisions have a big impact on the size of the feasible solutions area, which adds another layer of difficulty to the TD-IRP.

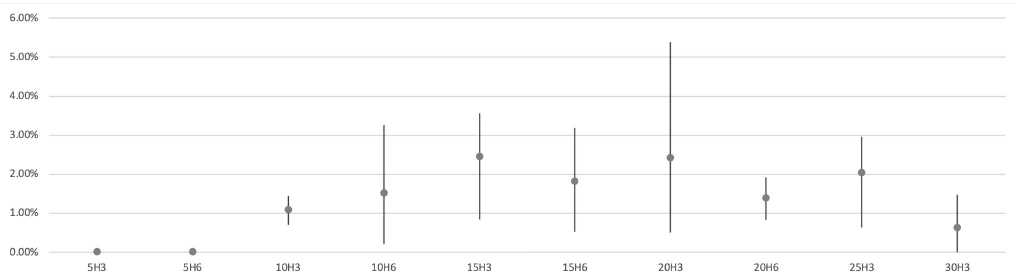
Observation 2: *Solutions of constant travelling time functions make good solutions in a time-dependent environment.* As we can see from column $g^{z}_{z_{|\mathcal{M}|=1}}$, the maximal gap between z and $z_{|\mathcal{M}|=1}$ is of 4.42%.

Observation 3: *Observation 2 should be nuanced.* Indeed, although the solutions of constant travelling time functions in a time-dependent environment are indisputably good, they can be nuanced with the following considerations:

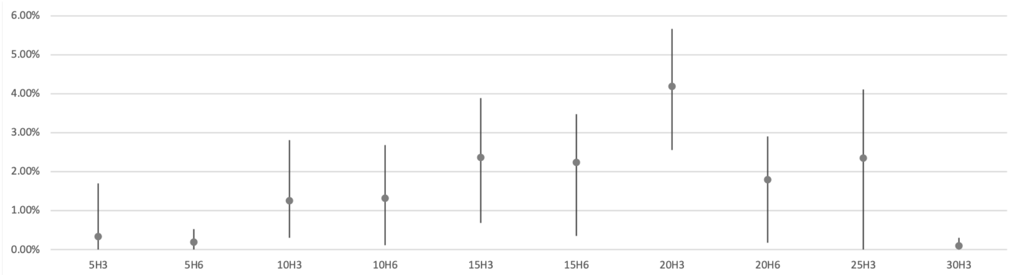
- the values of $g^{z}_{z_{|\mathcal{M}|=1}}$ are average values over only 5 values, which can be misleading if the standard deviation is high enough. Figure 7 shows the interval of distribution of these gaps through a line where in one end, the minimum is displayed, on the other end, the maximum and the dot represent the average values on all instance sets for each time step. We can see from the figure that gaps can go up to 8.2% when $|\mathcal{V}'|$=20 and $|\mathcal{M}| = 120$.
- As stated in the first observation, the TD-IRP is difficult to solve. Therefore, for instances when $|\mathcal{V}'| \geq 25$, the values of z are not optimal and have gaps up to 8.21% to the best lower bound found. On the other hand, the values of $z_{|\mathcal{M}|=1}$ are mostly optimal and have a maximal gap to the best lower bound of 0.24%. Therefore, to have a better idea of how the gap is evolving, a comparison between $z_{|\mathcal{M}|=1}$ and the best lower bound found when solving the time-dependent problem z^{LB} is needed. Figure 8 presents the distribution of these gaps. We can see from the figure that the gaps can go up to 13.63% when $|\mathcal{V}'| = 25$ and $|\mathcal{M}| = 120$.
- When looking closely at the structure of the optimal time-dependent solutions, we notice that the inventory levels of the clients and the supplier are not extremely different from the solutions of when $|\mathcal{M}| = 1$. Therefore, the gaps presented in Table 3 are mostly transportation costs, as the inventory costs are almost equal. In this case, the gain in transportation cost by optimising with time-dependent travelling time functions is more important than indicated in Table 3.
- Finally, in our model, we made the hypothesis that all tours start at the beginning of period p, i.e. at time step $m = 0$. This hypothesis can be very restrictive in a time-dependent environment as the departure time can have a big influence on



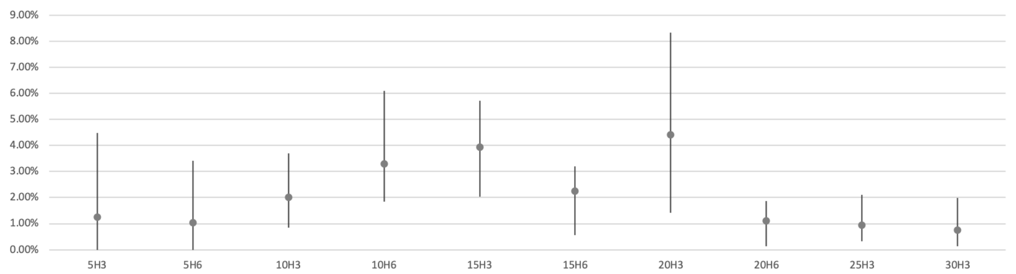
(a) $|\mathcal{M}| = 12$



(b) $|\mathcal{M}| = 30$

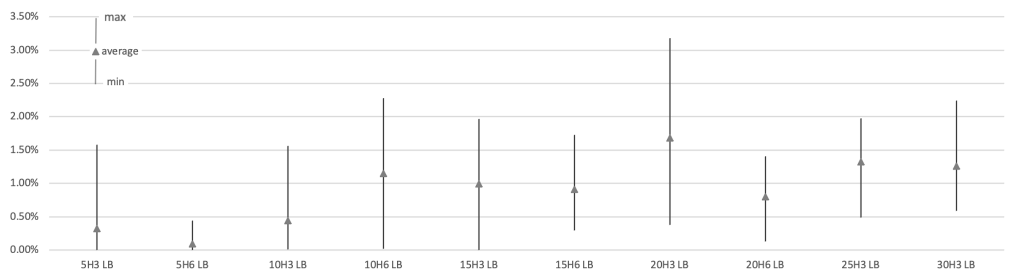


(c) $|\mathcal{M}| = 60$

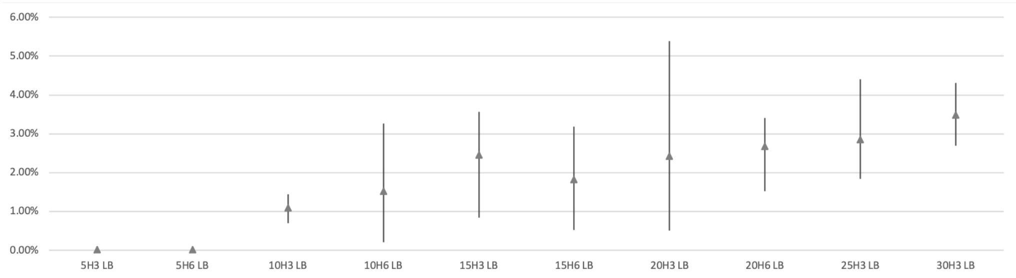


(d) $|\mathcal{M}| = 120$

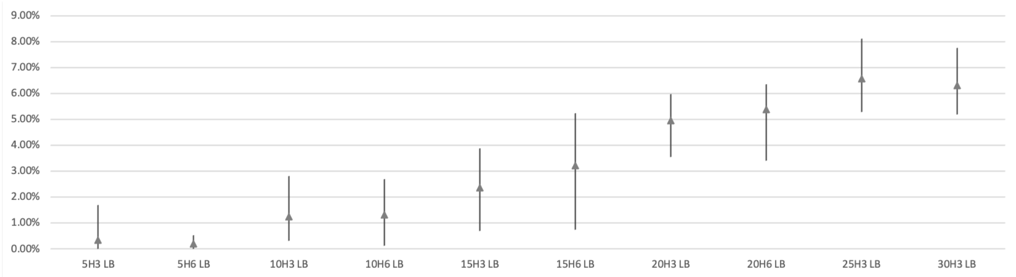
Figure 7: Distribution of $g^{z}_{z_{|\mathcal{M}|=1}}$



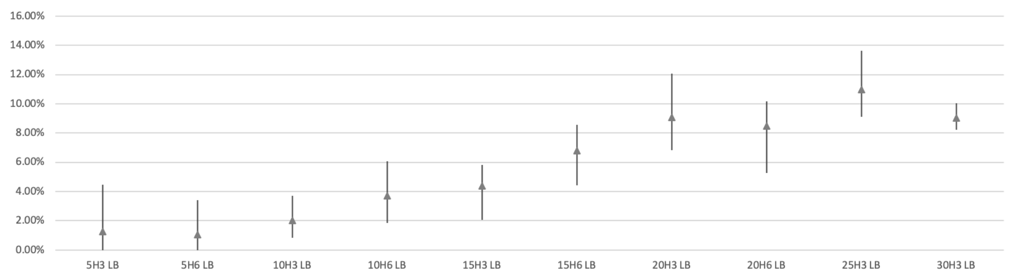
(a) $|\mathcal{M}| = 12$



(b) $|\mathcal{M}| = 30$



(c) $|\mathcal{M}| = 60$



(d) $|\mathcal{M}| = 120$

Figure 8: Distribution of $g_{z_{|\mathcal{M}|=1}}^{z^{LB}}$

the optimal tour.

For all these reasons, we believe that optimising with time-dependent functions can be beneficial in order to have cost-efficient solutions. However, solving large TD-IRP instances seems to be a computational challenge. Therefore, it is necessary to propose new algorithms to solve the TD-IRP more efficiently.

6 A matheuristic for the TD-IRP

In Section 5.3, we studied the structure of optimal TD-IRP solutions in comparison to solutions of constant travelling time functions when re-solved in a time-dependent environment. This comparison showed that the difference between the two solutions lies mostly in the sequence in which the clients are visited, and rarely in the set of clients visited at each period p or the quantities sent and inventory levels of said clients. Therefore, we propose a matheuristic to solve the TD-IRP by decomposing the problem into two parts: First, defining the inventory level, the quantities to send for each client and the clients to visit for each period. Second, defining the sequence in which the clients will be visited and the departure time steps from each location.

6.1 Matheuristic decomposition procedure

The results in Section 5 show that optimal constant travelling time solutions when re-solved in a time-dependent environment yield time-dependent solutions that differ from optimal time-dependent solutions, mostly in the routing component, and particularly the sequence of visiting the clients rather than the set of clients visited. Based on these observations, we propose a matheuristic that decomposes the problem into two parts: First, we define the inventory level, the quantities to send for each client and the clients to visit for each period by solving the problem using constant travelling time functions. Afterwards, for each period $p \in \mathcal{H}$, the sequence in which the clients are visited and the departure time steps from each location are defined by solving an independent TD-TSP. A mathematical formulation for the TD-TSP is presented below. $\mathcal{V}_p$ represents the set of locations to visit for each period p and $\mathcal{A}_p$ is a set of arcs linking them. Variables x_{ij} and x_{ij}^{m} represent the same variables as, respectively, x_{ij}^{p} and x_{ij}^{pm} – but the index of period p is no longer needed, as the TD-TSPs of each period are solved independently.

The full decomposition procedure is described in detail in Figure 9.

To optimally define the set of clients to be visited at each period $p \in \mathcal{H}$, the mathematical formulation presented in Section 3.1 is paired to the branch-and-cut procedure presented in Section 5.1. For the TD-TSP, the same branch-and-the cut procedure is used paired to the TD-TSP mathematical formulation presented above.

The choice to use the same algorithms and mathematical formulations to solve the

TD-TSP					
min	$obj = c \sum_{(i,j)\in\mathcal{A}_p} \sum_{m\in\mathcal{M}} f(i,j,m) \times x_{ij}^m$				
s.t	$\sum_{j\in\mathcal{V}_p \neq i} x_{ij}$	$=$	1	$\forall i \in \mathcal{V}_p$	(18)
	$\sum_{j\in\mathcal{V}_p \neq i} x_{ji}$	$=$	1	$\forall i \in \mathcal{V}_p$	(19)
	$\sum_{(i,j)\in\mathscr{S}} x_{ij}$	$\leq$	$\lvert\mathscr{S}\rvert - 1$	$\forall \mathscr{S} \subseteq \mathcal{A}_p$	(20)
	$\sum_{m\in\mathcal{M}} x_{ij}^m$	$=$	x_{ij}	$\forall (i,j) \in \mathcal{A}_p$	(21)
	$\sum_{j\in\mathcal{V}'_p} x_{0j}^0$	$=$	1		(22)
	$\sum_{v_k\in P\setminus\{v_n\}} \sum_{m_k\in T} x_{v_k,v_{k+1}}^{m_k}$	$\leq$	$\lvert P\rvert - 2$	$\forall [P,T]$ infeasible	(23)
	x_{ij}	$\in$	$\{0,1\}$	$\forall (i,j) \in \mathcal{A}_p$	(24)
	x_{ij}^m	$\in$	$\{0,1\}$	$\forall (i,j) \in \mathcal{A}_p,\ m \in \mathcal{M}'$	(25)

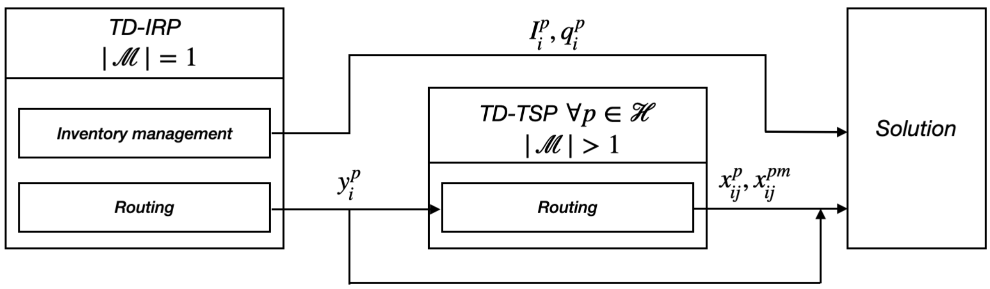


Figure 9: A description of the matheuristic procedure

decomposition process enables a fairer comparison between the exact approach and the matheuristic, in order to show more clearly the advantage of using such a decomposition. Indeed, better approaches in the literature exist that solve the TD-TSP more efficiently. But by using such state-of-the-art algorithms to solve the TD-TSP, the difference in performances would not be directly related to the decomposition approach but would instead be credited (at least partially) to the efficiency of the algorithms in question.

6.2 Numerical experiments

In order to efficiently compare the performances of the proposed matheuristic compared to the exact approach, both approaches will be solved in the same time limit of 3600 seconds. For the exact approach, and since there is only one branch-and-cut algorithm to solve, the Gurobi time limit parameter is used. However, for the matheuristic, for instances with $|\mathcal{H}| = 3$ and $|\mathcal{H}| = 6$, there are, respectively, 4 and 7 different models to solve: A TD-IRP when $|\mathcal{M}| = 1$, then independent TD-TSPs for each period $p \in \mathcal{H}$. Therefore, it is necessary to distribute the available computing time between these models in a way that ensures that all models are solved. In this case, the problem of defining the set of clients to visit is solved first with a 3600 seconds time limit, which is amply sufficient for this first step. The remaining time is then distributed equally between the different TD-TSPs. Afterwards, for each period $p \in \mathcal{H}$, if the TD-TSP is solved before its time limit, the remaining time is equally distributed on the not yet solved TD-TSPs. Algorithm 4 presents the full procedure of the matheuristic.

Algorithm 4: matheuristic for the TD-IRP

input: instance $\mathscr{I}$, a number of time steps M, an empty solution $\mathcal{S}$, and a time limit $TL = 3600$
solve $\mathscr{I}$ for $|\mathcal{M}| = 1$
fix the variables y_i^p, q_i^p and I_i^p of $\mathcal{S}$.
remaining time: $RT = TL - CPU_{S_{|\mathcal{M}|=1}}$
for $p \in \mathcal{H}$ **do**
    $TL_p^{\text{TD-TSP}} = \frac{RT}{|\mathcal{H}|}$
end for
for $p \in \mathcal{H}$ **do**
    a set of locations to visit $\mathcal{V}_p$ where $\mathcal{V}_p \subseteq \mathcal{V}$
    solve the TD-TSP for $\mathcal{V}_p$ with $|\mathcal{M}| = M$ and a time limit $TL_p^{\text{TD-TSP}}$
    remaining time from the TD-TSP of period p: $RT_p^{\text{TD-TSP}} = TL_p^{\text{TD-TSP}} - CPU_p^{\text{TD-TSP}}$
    if $RT_p^{\text{TD-TSP}} > 0$ **then**
        remaining set of unsolved TD-TSP $\mathcal{H}^{\text{remaining}} = \mathcal{H} \backslash \{1, ..., p\}$
        for $p' \in \mathcal{H}^{\text{remaining}}$ **do**
            $TL_{p'}^{\text{TD-TSP}} = TL_{p'}^{\text{TD-TSP}} + \frac{RT_p^{\text{TD-TSP}}}{|\mathcal{H}^{\text{remaining}}|}$
        end for
    end if
    set the values of variables x_{ij}^p and x_{ij}^{pm} for period p
end for
return $\mathcal{S}$

The results of the comparison are shown in Table 4 as follows: Columns $|\mathcal{H}|$, $|\mathcal{V}'|$, and $|\mathcal{M}|$ present, respectively, the length of the horizon, the number of clients in the network and the number of time steps of the travelling time function. Columns z, $g\%$, CPU and z^{LB} present, respectively, the objective value, the gap to the best lower bound found, the execution time and the best lower bound found for the exact approach. These values are average values over five instances of each combination of parameters $|\mathcal{H}|$, $|\mathcal{V}'|$ and $|\mathcal{M}|$. Columns z_{MH}, and CPU_{MH} represent, respectively, the objective value and the execution time, on average, of the matheuristic. Finally, columns $g^z_{z_{MH}}$, $g^{z^{LB}}_{z_{MH}}$ and $g^{CPU}_{CPU_{MH}}$ represent, respectively, the gap between z and z_{MH}, the gap between

z_{MH} and z^{LB} and finally the gap between CPU_{MH} and CPU where: $g^{z}_{z_{MH}} = \frac{z_{MH}-z}{z}$, $g^{z^{LB}}_{z_{MH}} = \frac{z_{MH}-z^{LB}}{z^{LB}}$ and $g^{CPU}_{CPU_{MH}} = \frac{CPU_{MH}-CPU}{CPU}$. The values indexed with an asterisk represent the ones for which at least one TD-TSP is not optimally solved.

6.3 Discussion

The results of Table 4 show that the matheuristic performs very well. These performances are discussed first in comparison to the best solutions found by the exact approach in Section 5.2. In a second phase, they are compared to the best lower bounds found by the exact approach.

For all instances solved optimally with the exact approach, the maximal average gap between z_{MH} and z is of 0.63%, when $|\mathcal{H}| = 3$, $|\mathcal{V}'| = 15$ and $|\mathcal{M}| = 60$. Figure 10 presents a more detailed look at how $g^{z}_{z_{MH}}$ is distributed. As we can see in Figure 10c, the largest gap for instances $|\mathcal{H}| = 3$, $|\mathcal{V}'| = 15$ and $|\mathcal{M}| = 60$ is of 1.51% and one instance is solved optimally. Now for all instances, the maximal gap goes up to 2.47% when $|\mathcal{V}'| = 30$ and $|\mathcal{M}| = 120$ (Figure 10d) whereas the minimum gap goes down to -3.57% when $|\mathcal{V}'| = 30$ and $|\mathcal{M}| = 60$ (Figure 10c). This means that the matheuristic is able to improve the best solution found by the exact approach within the time limit. Moreover, it does so in a shorter time, as shown by the gap in time $g^{CPU}_{CPU_{MH}}$ which in this case is of -35.33%.

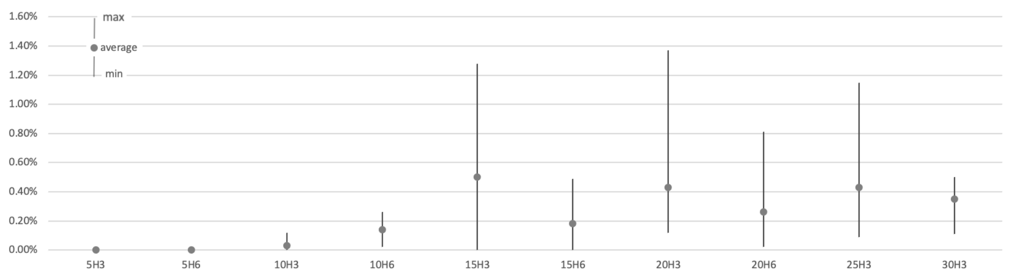
Since not all instances can optimally be solved with the exact approach, we compare the performances of the matheuristic to the best lower bounds found by the exact approach in order to have a more accurate idea of how it is performing. Figure 11 presents a more detailed look at the distribution of $g^{z^{LB}}_{z_{MH}}$. The largest gap $g^{z^{LB}}_{z_{MH}}$ of the instances for which all the TD-TSPs in the matheuristic are solved optimally, i.e. the ones not indexed with an asterisk in Table 4, is observed for instances where $|\mathcal{H}| = 6$, $|\mathcal{V}'| = 20$ and $|\mathcal{M}| = 120$. The average gap is of 5.12% whereas the maximal and minimal gaps are of, respectively, 6.29% and 2.68%. For instances for which all the TD-TSPs are not optimally solved, e.g. $|\mathcal{V}'| = \{25, 30\}$ and $|\mathcal{M}| = 120$, the maximal gap goes up to 11.45%. However, for these instances, the gaps can be reduced by improving the TD-TSPs that are not optimal, as the largest gap of these TD-TSPs to their best lower bound can go up to 22%. This can be done by exploiting the literature of TD-TSP which is getting increasingly richer, in comparison to the literature of TD-IRP which is sparse.

Moreover, it is worth noting that the values of z^{LB} are obtained using a branch-and-cut procedure. Therefore, the lower bound solutions have a high probability of containing sub-tours and time-dependent infeasible paths. In this context, the gaps $g^{z^{LB}}_{z_{MH}}$ that are observed when TD-IRPs are not optimally solved can be nuanced, as z^{LB} can be improved, which makes the real gap between z_{MH} and the optimal TD-IRP solutions even closer than $g^{z^{LB}}_{z_{MH}}$.

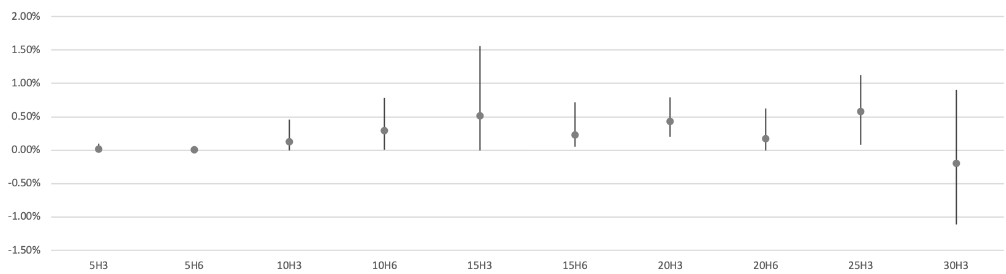
Although we cannot generalise these results to all instances but only to the benchmark

Table 4: Results

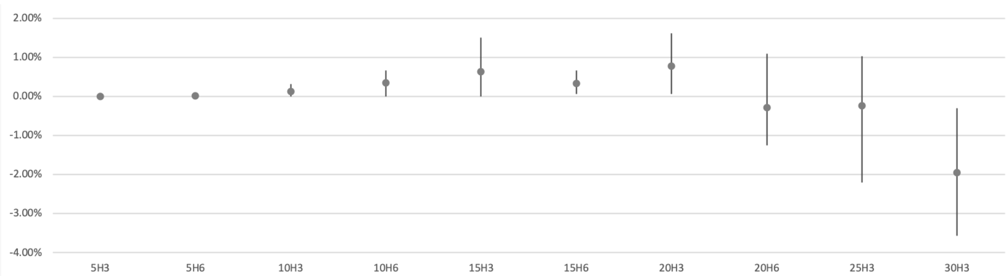
$\lvert\mathcal{H}\rvert$	$\lvert\mathcal{V}'\rvert$	$\lvert\mathcal{M}\rvert$	z	$g\%$	CPU	z^{LB}	z_{MH}	CPU_{MH}	$g^{z}_{z_{MH}}\%$	$g^{z^{LB}}_{z_{MH}}\%$	$g^{CPU}_{CPU_{MH}}\%$
		1	3790.27	0.00	0.00						
		12	3387.27	0.00	0.00	3387.27	3387.27	0.01	0.00	0.00	0.00
	5	30	3136.02	0.00	0.01	3136.02	3136.47	0.01	0.02	0.02	0.00
		60	2916.07	0.00	0.02	2916.07	2916.07	0.02	0.00	0.00	-15.00
		120	2584.09	0.00	0.05	2584.09	2597.47	0.03	0.57	0.57	-37.67
		1	5544.34	0.00	0.04						
		12	4906.04	0.00	0.06	4906.04	4907.54	0.07	0.03	0.03	25.43
	10	30	4579.25	0.00	0.55	4579.25	4584.74	0.22	0.13	0.13	-59.89
		60	4183.26	0.00	1.10	4183.26	4188.54	0.46	0.13	0.13	-51.96
		120	3752.34	0.00	45.97	3752.34	3756.54	1.14	0.12	0.12	-76.48
		1	6822.30	0.00	0.07						
		12	6169.50	0.00	0.32	6169.50	6198.90	0.40	0.50	0.50	37.14
	15	30	5823.55	0.00	4.46	5823.55	5852.10	1.14	0.51	0.51	-68.28
		60	5346.01	0.00	70.24	5346.01	5377.70	2.68	0.63	0.63	-94.91
		120	4701.56	0.45	1610.18	4681.23	4740.70	7.73	0.85	1.31	-97.62
		1	8588.07	0.00	0.28						
3		12	7699.65	0.00	4.87	7699.65	7734.47	1.29	0.43	0.43	-40.81
	20	30	7198.26	0.00	782.55	7198.10	7230.07	6.25	0.43	0.43	-96.85
		60	6622.56	0.75	2240.37	6574.74	6675.07	10.63	0.78	1.53	-98.64
		120	6025.68	4.47	3600.03	5756.28	6035.87	74.80	0.23	4.70	-97.92
		1	9246.05	0.00	0.78						
		12	8426.63	0.00	160.87	8426.63	8463.05	8.23	0.43	0.43	-88.87
	25	30	7864.11	0.80	2308.50	7798.68	7909.45	70.73	0.58	1.38	-96.03
		60	7378.17	4.16	3600.03	7063.85	7357.05	826.04	-0.24	3.88	-77.05
		120	6807.12	9.92	3600.04	6132.34	6649.05*	2153.25	-2.30*	7.39*	-40.19
		1	10611.99	0.00	0.73						
		12	9615.59	0.29	2397.94	9587.89	9648.99	79.33	0.35	0.64	-96.57
	30	30	9160.58	2.85	3600.02	8901.15	9143.19	1035.16	-0.20	2.64	-71.25
		60	8599.19	6.21	3600.03	8062.89	8427.79	1351.96	-1.95	4.13	-62.45
		120	7761.99	8.21	3600.10	7122.38	7752.59*	3600.03	-0.11*	8.10*	0.00
		1	8385.85	0.00	0.04						
		12	7514.45	0.00	0.03	7514.27	7514.45	0.04	0.00	0.00	68.33
	5	30	6941.13	0.00	0.07	6940.97	6941.85	0.05	0.01	0.01	0.48
		60	6448.17	0.00	0.21	6448.17	6449.45	0.10	0.02	0.02	-32.19
		120	5740.21	0.00	0.26	5740.00	5741.25	0.12	0.02	0.02	-43.81
		1	11040.83	0.00	0.11						
		12	9856.17	0.00	0.27	9855.96	9870.23	0.22	0.14	0.14	5.65
	10	30	9209.18	0.00	13.60	9209.18	9235.43	0.77	0.29	0.29	-86.55
		60	8496.00	0.00	47.21	8495.84	8525.23	1.35	0.35	0.35	-93.95
		120	7532.50	0.39	1074.80	7505.65	7578.83	3.16	0.64	1.02	-96.22
6		1	13923.93	0.00	0.47						
		12	12564.94	0.00	3.05	12564.94	12586.33	1.15	0.18	0.18	-51.14
	15	30	11923.00	0.00	495.22	11922.77	11948.73	3.56	0.23	0.23	-96.89
		60	10912.59	0.95	2929.27	10812.31	10947.33	7.25	0.33	1.28	-99.00
		120	9760.72	4.45	3600.02	9324.18	9721.13	26.58	-0.40	4.02	-99.26
		1	17726.17	0.00	2.16						
		12	15905.55	0.02	1143.01	15902.29	15947.17	5.84	0.26	0.28	-94.54
	20	30	14821.00	1.28	3600.01	14636.60	14846.97	17.90	0.17	1.45	-99.50
		60	13695.51	3.52	3600.03	13216.30	13654.37	48.41	-0.28	3.22	-98.66
		120	12501.52	7.31	3600.04	11593.15	12251.17	205.82	-2.03	5.12	-94.28



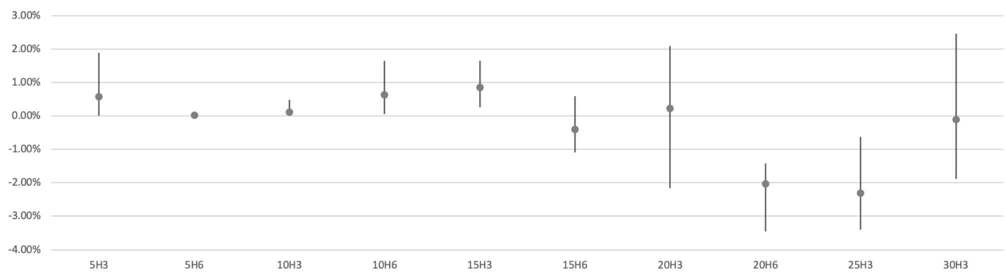
(a) $|\mathcal{M}| = 12$



(b) $|\mathcal{M}| = 30$

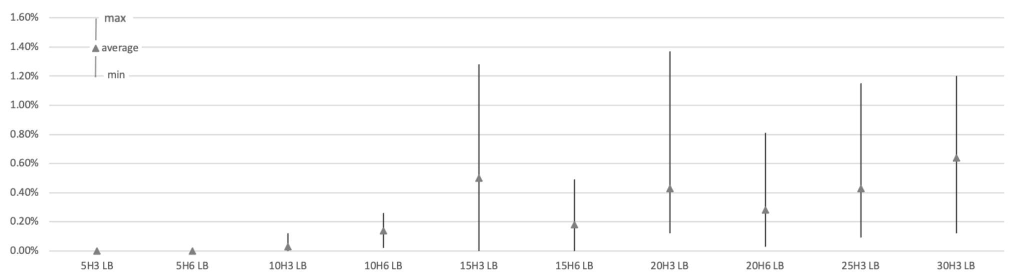


(c) $|\mathcal{M}| = 60$

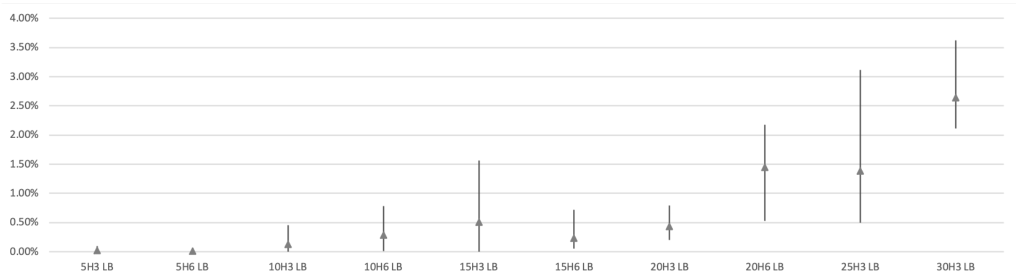


(d) $|\mathcal{M}| = 120$

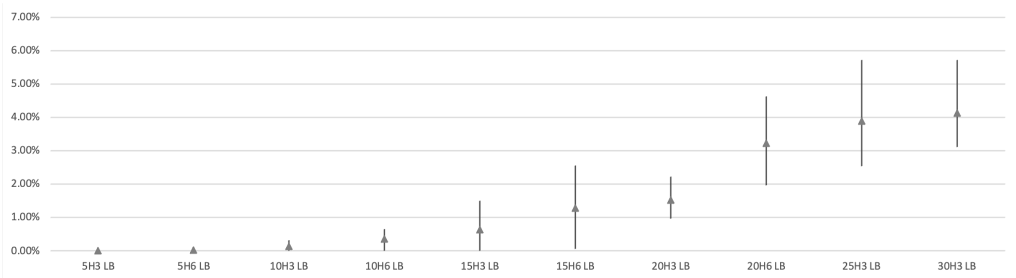
Figure 10: Distribution of $g^{z}_{z_{MH}}$



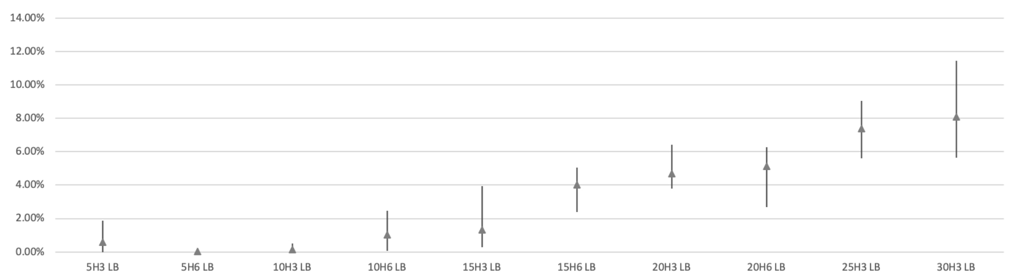
(a) $|\mathcal{M}| = 12$



(b) $|\mathcal{M}| = 30$



(c) $|\mathcal{M}| = 60$



(d) $|\mathcal{M}| = 120$

Figure 11: Distribution of $g_{z_{MH}}^{z_{LB}}$

used in this paper, intuitively speaking, if we are comparing an IRP with a TD-IRP with two travelling time functions that are generated from the same data and the only difference between them is the time granularity, it is not unreasonable to imagine that this difference will not amount in an extreme change in the clustering problem, i.e. which clients to visit for each period. However, changes in the sequence of visiting these clients is highly expected. Therefore, the difference in the objective values of the IRP and the TD-IRP would mostly be the result of transportation cost.

As a conclusion, the decomposition procedure of our matheuristic proves to be very efficient. The strengths of this matheuristic lie in the fact that it is inspired by observations of optimal TD-IRP solutions. Moreover, it has the capacity of scaling as its improvement depends on the efficiency of the formulations and algorithms of IRP and TD-TSP, on which the literature is exponentially increasing.

7 Conclusions & perspectives

This paper considers a variant of the IRP, the TD-IRP. In this variant, the travelling time does not only depend on the departure and arrival locations, but on the time of departure as well. In order to include the time-dependent aspect to the IRP, a review of the TD-IRP is conducted. The review shows that the literature of TD-IRP is rather sparse and only one paper handles it. As a result, we turn to pure time-dependent routing problems. The rich literature of time-dependent routing problems shows that these problems are harder to solve than their basic counterparts, thus the need for new efficient approaches in order to solve real-sized instances. Moreover, it gives insight on how to incorporate the time-dependent aspect for the IRP and benchmarks for experimentation.. Inspired by this literature, a mathematical formulation for the TD-IRP is proposed and a new benchmark is generated. This benchmark is based on benchmarks from the literature of IRP and TD-TSP in order to provide more realistic instances for the community of researchers. In order to investigate the impact of inventory decisions on the structure of optimal time-dependent tours, a branch-and-cut procedure is proposed and a comparison between optimal TD-IRPs and optimal IRPs computed in a time-dependent environment is carried out. The results show that optimising in a time-dependent environment can be beneficial, cost-wise. Based on the observations made on these results, a matheuristic that decomposes the problem into a problem of defining the set of clients to visit for each period first and routing second, is proposed. The results show that the proposed matheuristic is very efficient. Furthermore, its performances can be improved by taking advantage of the rich literature of time-dependent routing problems in comparison to the sparse literature of TD-IRP.

Future works on this topic can take advantage of the time-dependent environment by considering the departure time as a decision variable rather than imposing it at the beginning of the period. Such a hypothesis can have a huge impact on the improvement of the cost of the time-dependent solutions, as it is sometimes more efficient to leave later in order to avoid congestion. Another perspective would be to extend the problem to the TD-IRP with time windows, since waiting at nodes can be of great relevance in this context. To solve the problem more efficiently, we can take advantage of the rich literature of TD-TSPs to improve the matheuristic performances by implementing or proposing new exact approaches such as dynamic programming for instances up to 30 clients, or the dynamic discretisation discovery framework proposed by the authors of Minh Vu et al. (2018) for larger instances. In parallel, it is necessary to propose new formulations or valid inequalities for the TD-IRP that provide tighter lower bounds in order to efficiently assess the performances of the matheuristic. Finally, another perspective would be to consider that the service time is no longer constant but can depend on the time of visit and the quantity that needs to be unloaded. This can be very important for products that require a long unloading time, such as fuel in gas stations.

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Highlights

- A MILP formulation for the inventory routing problem with time-dependent travelling times
- A novel TD-IRP benchmark based on real-life routing data
- Showing the relevance of optimising the IRP with time-dependent travelling times
- A matheuristic determining clients to visit and quantity to deliver first and routing second
- The matheuristic proves very efficient with small gaps to the best lower bounds found